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Uniqueness and Multiplicity for Perturbations of the Yamabe Problem on S^n

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SUMMARY. - *Motivated by an uniqueness result for linear perturbations with constant coefficients of the conformal laplacian on the sphere, we investigate, via a finite dimensional reduction, more general perturbations of the conformal laplacian, exibiting cases in which uniqueness fails*

1. Introduction

In this paper we study the equation

$$-\triangle_h u + \lambda(x) u = u^{\frac{n+2}{n-2}} \tag{1}$$
$$u \in H_1^2(S^n) \qquad u > 0$$

where (S^n, h) is the n-dimensional sphere equipped with the standard metric.
When $\lambda(x) \equiv \frac{n(n-2)}{4}$, the solutions are explicitely known and they are obtained by the action of the conformal group on the constant solution

$$\bar{u} := [\frac{n(n-2)}{4}]^{\frac{n-2}{4}}$$

In contrast, if $\lambda(x) \equiv \lambda < \frac{n(n-2)}{4}$, as a consequence of a remarkable result by Bidaut-Veron and Veron, see [4], (1) admits just the constant solution.

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The main purpose of this paper is to show that uniqueness fails for $|\epsilon|$ small if $\lambda(x) = \frac{n(n-2)}{4} + \epsilon f(x)$ and f changes sign.
More precisely, we will prove

THEOREM 1.1. *Let* $\lambda_\epsilon(x) = \frac{n(n-2)}{4} + \epsilon f(x) + g(\epsilon, x)$, *with* $\| g(\epsilon, x) \|_\infty = o(\epsilon)$ *as* $\epsilon \to 0$, *be in* $C^0(S^n)$. *Then (1) is solvable for* $|\epsilon|$ *small. Furthermore, (1) has at least two solutions if* f *changes sign and* $n \geq 4$ *or if* $n = 3$ *and* $\int_{S^n} f = 0$ *with* $f \neq 0$.

A related problem in R^n has been considered in [1].
We wish to thank Prof. Emmanuel Hebey and his whole research group for their kind hospitality and for the helpful suggestions received in the period of study spent in the Cergy-Pontoise University.
After this paper was finished, we learned from Prof. Ambrosetti that in a recent paper of Cingolani similar results are obtained.

2. A finite dimensional reduction

Weak solutions of (1) are critical points of the energy functional

$$E_\epsilon(u) := E_0(u) + G(\epsilon, u) \ , \ \ u \in H_1^2(S^n)$$

where

$$E_0(u) := \frac{1}{2} \int_{S^n} | \nabla u |^2 \, dv(h)$$

$$+ \frac{n(n-2)}{8} \int_{S^n} u^2 dv(h) - \frac{n-2}{2n} \int_{S^n} (u_+)^{\frac{2n}{n-2}} dv(h)$$

$$G(\epsilon, u) := \frac{1}{2} \int_{S^n} (\epsilon f + g(\epsilon, x)) u^2 dv(h)$$

The peculiar property of E_0 is its conformal invariance.
Let us consider, in particular, the following conformal transformations

$$\varphi_{\sigma,t}(y) = \pi_\sigma^{-1}(t \pi_\sigma(y)) \qquad \sigma \in S^n \ , \ \ t \geq 1$$

where π_σ denotes the stereographic projection from σ as the north pole.
They act on $H_1^2(S^n)$ through the following isomorphisms

$$T_{\sigma,t} u(y) := (u \circ \varphi_{\sigma,t})(y) \mid \det d\varphi_{\sigma,t}(y) \mid^{\frac{n-2}{2n}}$$

and

$$E_0(T_{\sigma,t}u) = E_0(u) \qquad \forall(\sigma,t) \in S^n \times [1,+\infty), \;\; \forall u \in H_1^2(S^n)$$

From the conformal invariance, it easily follows that $\nabla E_0(T_{\sigma,t}u) = 0$ if $\nabla E_0(u) = 0$.
In particular, if $\bar{u} \equiv [\frac{n(n-2)}{4}]^{\frac{n-2}{4}}$ denotes the constant solution,

$$Z := \{T_{\sigma,t}\bar{u} = \bar{u} \mid \det d\varphi_{\sigma,t}(y) \mid^{\frac{n-2}{2n}}\} \tag{2}$$

is a *critical manifold* for E_0. Actually, it can be shown that these are all the critical points of E_0.
A proof of this claim in book form can be found in [7]. References where such constructions are used are [6] and [5]. See also [2].
The critical points of E_ϵ can be found as critical points of E_ϵ constrained to some manifold Z_ϵ close to Z, following the same perturbation technique used in [1].
Let a family of C^2 functionals $\{E_\epsilon\}$ be defined on a Hilbert space E of the form

$$E_\epsilon(u) = E_0(u) + G(\epsilon, u) \tag{3}$$

We assume that the unperturbed functional E_0 satisfies the following assumptions:
$(A1)$ E_0 possesses a finite dimensional manifold Z of critical points at a fixed energy level b, that will be called *critical manifold*
$(A2)$ $D^2E_0(z)$ is a Fredholm map with index 0 $\forall z \in Z$
$(A3)$ $T_zZ = Ker(D^2E_0(z))$ $\forall z \in Z$ (T_zZ denotes the tangent space to Z at z)
$(B1)$ there exist $\alpha > 0$ and a continuous function $\Gamma : Z \to R$ such that uniformly on compact subsets of Z

$$\Gamma(z) = \lim_{\epsilon \to 0} \frac{G(\epsilon, z)}{\epsilon^\alpha}$$

$$G'(\epsilon, z) = o(\epsilon^{\frac{\alpha}{2}})$$

Under the preceding assumptions, one can use the Implicit Function Theorem to show for $\mid \epsilon \mid$ small the existence of $w = w(\epsilon, z) \in (T_zZ)^\perp$ such that

$$E'_\epsilon(z + w) \in T_zZ$$

$$\|w(\epsilon,z)\| = o(\epsilon^{\frac{\alpha}{2}})$$

uniformly on compact subsets of Z.
Letting $Z_\epsilon = \{z+w(\epsilon,z)\}$, we obtain a manifold locally diffeomorphic to Z such that for $|\epsilon|$ small any critical points of E_ϵ restricted to Z_ϵ is a stationary point of E_ϵ.
From the development of E_ϵ on Z_ϵ

$$E_\epsilon |_{Z_\epsilon} (u) = E_\epsilon(z+w(\epsilon,z)) = E_0(z)+(E_0'(z) \mid w(\epsilon,z))+O(\|w(\epsilon,z)\|^2)+$$

$$+G(\epsilon,z) + (G\,'(\epsilon,z) \mid w(\epsilon,z)) + O(\|w(\epsilon,z)\|^2) = b + \epsilon^\alpha\Gamma(z) + o(\epsilon^\alpha)$$

uniformly on compact subsets of Z, one can derive a general existence result, wich we will use in a particular case.

THEOREM 2.1. *Let $E_\epsilon \in C^2(E,R)$ be of the form (3), where E_0 and $G(\epsilon,u)$ satisfy $A1,2,3$ and $B1$ and suppose that there exists a critical point $\bar z \in Z$ of Γ such that one of the following conditions holds*
(i) $\bar z$ is nondegenerate
(ii) $\bar z$ is a strict local minimum or maximum
(iii) $\bar z$ is isolated and the local topological degree of Γ' at $\bar z$, $deg_{loc}(\Gamma',0)$, is different from zero. Then for $|\epsilon|$ small enough, the functional E_ϵ has a critical point u_ϵ such that $u_\epsilon \to \bar z$ as $\epsilon \to 0$.

It is easy to check that all the assumptions are satisfied in our case; in particular, Z, given as in (2), is a smooth manifold diffeomorphic to B^{n+1}, the open unit ball in R^{n+1}, through the map $\xi = \rho\sigma \in B^{n+1} \to T_{\sigma,(1-\rho)^{-1}}\bar u$.
Furthermore, the nondegeneracy assumption $A3$ is satisfied, because the kernel of the linearized operator at $\bar u$ is the eigenspace of the Laplace-Beltrami operator corresponding to the first eigenvalue $\lambda_1 = n$, wich has dimension $n+1$, see [3], while by conformal invariance

$$ker D^2E_0(T_{\sigma,t}\bar u) = T_{\sigma,t} ker D^2 E_0(\bar u)$$

Finally, we can take $\alpha = 1$ and get

$$\Gamma(\rho\sigma) = \frac{1}{2}c_n^2 \int_{S^n} f(y) \mid \det d\varphi_{\sigma,(1-\rho)^{-1}} \mid^{\frac{n-2}{n}} dv(y) \qquad (4)$$

where $c_n = [\frac{n(n-2)}{4}]^{\frac{n-2}{4}}$.

3. Proof of Theorem 1.1

We begin with an expansion around boundary points of the function Γ.

LEMMA 3.1. *If $n \geq 5$, for every $\sigma \in S^n$ it results*

$$\Gamma(\rho\sigma) = c_n^2 2^{n-1}(1-\rho)^2[f(-\sigma)\int_{R^n}(1+\mid x\mid^2)^{-(n-2)}dx+o(1)] \quad as \ \rho \to 1$$

If $n = 4$, for every $\sigma \in S^n$ it results

$$\Gamma(\rho\sigma) = -8\omega_3 c_4^2(1-\rho)^2 \ln(1-\rho)[f(-\sigma)+o(1)] \ \ as \ \ \rho \to 1$$

If $n = 3$, for every $\sigma \in S^n$, it results

$$\Gamma(\rho\sigma) = c_3^2(1-\rho)[\int_{S^3}\frac{f(y)}{1+\cos d(\sigma,y)}dv(y)+o(1)] \ \ as \ \ \rho \to 1$$

where

$$c_n = [\frac{n(n-2)}{4}]^{\frac{n-2}{4}}$$

Proof. We have (see [7]) that

$$\mid \det d\varphi_{\sigma,t}\mid^{\frac{n-2}{n}}(y) = (t\frac{1+\mid \pi_\sigma(y)\mid^2}{1+t^2\mid \pi_\sigma(y)\mid^2})^{n-2}$$

so that, integrating in stereographic coordinates

$$\int_{S^n} f(y)\mid \det d\varphi_{\sigma,t}\mid^{\frac{n-2}{n}}(y)dv(y) =$$

$$= 2^n\int_{R^n}\frac{f(\pi_\sigma^{-1}(x))t^{n-2}}{(1+t^2\mid x\mid^2)^{(n-2)}(1+\mid x\mid^2)^2}dx =$$

$$= 2^n t^{-2}\int_{R^n}\frac{(f\circ\pi_\sigma^{-1})(\frac{x}{t})}{(1+\mid x\mid^2)^{(n-2)}(1+\mid \frac{x}{t}\mid^2)^2} =$$

$$= 2^n t^{-2}[f(-\sigma)\int_{R^n}(1+\mid x\mid^2)^{-(n-2)}dx+$$

$+o(1)]$ as $t=(1-\rho)^{-1}\to+\infty$
by dominated convergence, if $n\geq 5$.
If $n=4$, it is enough to split the integral into two pieces:

$$I_1:=16t^{-2}\int_{|x|\geq 1}\frac{(f\circ\pi_\sigma^{-1})(x)}{(t^{-2}+\mid x\mid^2)^2(1+\mid x\mid^2)^2}dx=$$

$$=16t^{-2}[\int_{|x|\geq 1}\frac{(f\circ\pi_\sigma^{-1})(x)}{\mid x\mid^4(1+\mid x\mid^2)^2}dx+o(1)]$$

and

$$I_2:=16t^{-2}\int_{|x|\leq t}\frac{(f\circ\pi_\sigma^{-1})(\frac{x}{t})}{(1+\mid x\mid^2)^2(1+\mid\frac{x}{t}\mid^2)^2}dx=$$

$$=16t^{-2}(f(-\sigma)+o(1))\omega_3\int_0^t\frac{r^3}{(1+r^2)^2}dr=$$

$$=16\omega_3\frac{\ln t}{t^2}[f(-\sigma)+o(1)]$$

Finally, if $n=3$, exactly as for I_1, we now get

$$8\int_{R^3}\frac{(f\circ\pi_\sigma^{-1})(x)t}{(1+t^2\mid x\mid^2)(1+\mid x\mid^2)^2}dx=$$

$$=8t^{-1}[\int_{R^3}\frac{(f\circ\pi_\sigma^{-1})(x)}{(1+\mid x\mid^2)^2\mid x\mid^2}dx+o(1)]$$

Since $\pi_\sigma^{-1}(\frac{z}{|z|^2})=\pi_{-\sigma}^{-1}(z)$, after the change of variable $x=\frac{z}{|z|^2}$, the integral above rewrites

$$\int_{R^3}(f\circ\pi_\sigma^{-1})(\frac{z}{\mid z\mid^2})(1+\mid z\mid^2)^{-2}dz=$$

$$=2^{-3}\int_{R^3}(f\circ\pi_{-\sigma}^{-1})(z)(\frac{2}{1+\mid z\mid^2})^3(1+\mid z\mid^2)dz=$$

$$=2^{-3}\int_{S^3}f(y)(1+\mid\pi_{-\sigma}(y)\mid^2)dv(y)=2^{-2}\int_{S^3}\frac{f(y)}{1+\cos d(\sigma,y)}dv(y)$$

because

$$1+\mid\pi_{-\sigma}(y)\mid^2=1+\frac{1-y_{n+1}^2}{(1+y_{n+1})^2}=\frac{2}{1+y_{n+1}}\ \text{ and }\ y_{n+1}=\cos d(\sigma,y)$$

□

Proof. (Theorem 1.1)
If $n \geq 4$, we see from the Lemma that Γ vanishes at ∂B^{n+1} and changes sign with f around boundary points. Hence Γ has a positive maximum and a negative minimum and so it does $E_\epsilon |_{Z_\epsilon}$.
Existence of at least one solution, wich is not necessarily a minimum, follows similarly, without sign assumption on f.
Finally, if $n = 3$, from

$$\int_{S^3} d\sigma (\int_{S^3} \frac{f(y)}{1 + \cos d(\sigma, y)} dy) = \int_{S^3} f(y) (\int_{S^3} \frac{d\sigma}{1 + \cos d(\sigma, y)}) dy$$

and the independence on y of $\int_{S^3} \frac{d\sigma}{1+\cos d(\sigma,y)} > 0$ because of the symmetry, we see that $\sigma \to \int_{S^3} \frac{f(y)}{1+\cos d(\sigma,y)} dy$ has to change sign if $\int_{S^3} f = 0$.
So, again, we get a positive maximum and a negative minimum.
Hence, from Theorem 2.1, for ϵ small, we find solutions u_ϵ of the equation

$$-\triangle_h u + \lambda_\epsilon(x) u = (u_+)^{\frac{n+2}{n-2}}$$

If ab absurdo u_ϵ is not nonnegative, then there exists y_0 an absolute minimum point of u_ϵ on S^n with $u_\epsilon(y_0) < 0$.
So

$$\lambda_\epsilon(y_0) u_\epsilon(y_0) = (u_\epsilon)_+(y_0)^{\frac{n+2}{n-2}} + \triangle_h u_\epsilon(y_0) \geq 0$$

Then, for ϵ small, $u_\epsilon(y_0) \geq 0$, a contradiction.
The function u_ϵ must be nonnegative and from the maximum principle either vanishes or is positive.
But u_ϵ cannot vanish identically for ϵ small because its energy level is near $b > 0$.

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