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Complex Foliations in Generalized Twistor Spaces

M. Migliorini and A. Tomassini (*)

Summary. - *We consider a natural almost complex distribution on the associated bundle* $F^{(n)}(M)$ *to the principal bundle of the g-orthogonal oriented frames on a Riemannian manifold* (M, g)*, with standard fibre* $\frac{\mathrm{SO}(2n+k)}{\mathrm{U}(n)\times \mathrm{SO}(k)}$*: we find necessary and sufficient conditions ensuring that the distribution is an almost complex foliation in* $F^{(n)}(M)$ *and we compute the Nijenhuis tensor. Finally, we characterize the local sections of* $F^{(n)}(M)$*.*

0. Introduction

The study of the Twistor Space $Z(M)$ of a Riemannian manifold M has a fundamental role in differential geometry. For an overview in twistor geometry we refer to [1], [2], [3], [4] while, for some observations on the natural almost complex structures $\mathbb{J}$ of $Z(M)$ we may see [5], [6].

A natural generalization of the Twistor Space over a Riemannian manifold was introduced by Rawnsley and Salamon (see [7], [8]). They investigated about the holomorphic and harmonic maps into

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(*) Authors' addresses: Massimiliano Migliorini, Dipartimento di Matematica "Ulisse Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy; e-mail: max@alibaba.math.unifi.it
Adriano Tomassini, Dipartimento di Matematica ed Applicazioni, Università di Palermo, Viale delle Scienze, 90128 Palermo, Italy; e-mail: adriano@ipamat.math.unipa.it
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the generalized Twistor Space. This generalization is obtained as the associated bundle to $SO_g(M)$, the principal bundle of the positively oriented orthonormal frames over a $2n+k$-dimensional Riemannian manifold (M,g), $F^{(n)}(M) := SO_g(M) \times_{\mathrm{SO}(2n+k)} \frac{\mathrm{SO}(2n+k)}{\mathrm{U}(n)\times\mathrm{SO}(k)}$. A section of $F^{(n)}(M)$ gives rise to a f-*structure* in the sense of Yano (see [9]), i. e. a $(1,1)$ tensor f on M satisfying $f^3+f=0$.

In this paper we are interested to the existence of complex foliations of $F^{(n)}(M)$ and to describe sections of $F^{(n)}(M)$.
In the first Section we start describing the standard fibre $\frac{\mathrm{SO}(2n+k)}{\mathrm{U}(n)\times\mathrm{SO}(k)}$, then we recall the construction of $F^{(n)}(M)$ and its natural f-structure, that gives rise to an almost complex distribution of $F^{(n)}(M)$, $D\oplus Z$. In Sections 2, 3 we give the explicit conditions to the involutivity of the distribution by the curvature form of the pricipal bundle $SO_g(M)$ and compute the Nijenhuis tensor of the almost complex structure on the leaves (see Theorem 2.2 and Lemma 3.2). In the last Section we describe the local sections of $F^{(n)}(M)$ (see Proposition 4.3).

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1. Main construction

Let $Z(n,k)$ be the homogeneous space given by $\frac{\mathrm{SO}(2n+k)}{\mathrm{U}(n)\times\mathrm{SO}(k)}$ with projection $p:\mathrm{SO}(2n+k)\to Z(n,k)$, i. e. $p(A)=p(B)$ if and only if there exists $C\in\mathrm{U}(n)\times\mathrm{SO}(k)$ such that $B=AC$. We have a splitting of the Lie algebra $\mathfrak{so}(2n+k)=\mathfrak{h}\oplus\mathfrak{m}$, with

$$\begin{aligned}
\mathfrak{h} &:= \mathfrak{u}(n)\oplus\mathfrak{so}(k)\\
\mathfrak{m} &:= \left\{\begin{pmatrix} X & Y\\ -{}^tY & 0\end{pmatrix}\in\mathfrak{so}(2n+k)\,|\,X\in\mathfrak{s}(n),\ Y\in M_{2n,k}(\mathbb{R})\right\}\\
\mathfrak{s}(n) &:= \{X\in\mathfrak{so}(2n)\,|\,XJ_n=-J_nX\}\\
J_n &= \begin{pmatrix} 0 & -I_n\\ I_n & 0\end{pmatrix}.
\end{aligned}$$

Note that $[\mathfrak{m},\mathfrak{h}]\subset\mathfrak{m}$ and therefore $Z(n,k)$ is *reductive* of dimension $n^2-n+2nk$, but not *symmetric* ($[\mathfrak{m},\mathfrak{m}]\not\subset\mathfrak{h}$).

By considering the natural action

$$\begin{array}{ccc} \mathrm{SO}(2n+k) \times \dfrac{\mathrm{SO}(2n+k)}{\mathrm{U}(n)\times\mathrm{SO}(k)} & \to & \dfrac{\mathrm{SO}(2n+k)}{\mathrm{U}(n)\times\mathrm{SO}(k)} \\ (A,X) & \mapsto & AX^tA \end{array}$$

the isotropy subgroup at $T = \begin{pmatrix} J_n & 0 \\ 0 & I_k \end{pmatrix}$ is $\mathrm{U}(n)\times\mathrm{SO}(k)$; then

$$Z(n,k) = \{AT^tA \mid A \in \mathrm{SO}(2n+k)\}\,.$$

We observe that the tangent space at $P \in Z(n,k)$ is

$$T_P Z(n,k) = \{XP - PX \mid X \in \mathfrak{so}(2n+k)\}\,.$$

Any point $P \in Z(n,k)$ represents an oriented $2n$-dimensional linear subspaces of $\mathbb{R}^{2n+k}$ equipped with a positively oriented orthogonal complex structure.

We have a natural application

$$\mu : Z(n,k) \to G_{2n,k} := \frac{\mathrm{SO}(2n+k)}{\mathrm{SO}(2n)\times\mathrm{SO}(k)}$$

from $Z(n,k)$ to the *Grassmannian* of the oriented $2n$-dimensional linear subspace of $\mathbb{R}^{2n+k}$. If we take $H = \begin{pmatrix} I_{2n} & 0 \\ 0 & -I_k \end{pmatrix}$, then the isotropy subgroup at H, for the natural action of $\mathrm{SO}(2n+k)$ is $\mathrm{SO}(2n)\times\mathrm{SO}(k)$; therefore $G_{2n,k} = \{AH^tA \mid A \in \mathrm{SO}(2n+k)\}$ and μ is defined by

$$AT^tA \overset{\mu}{\mapsto} AH^tA\,.$$

Then μ is a fibration with standard fibre $Z(n) := Z(n,0)$.
Let $P = AT^tA$ be in $Z(n,k)$, the tangent space at P to the fibre is

$$T_P\mu^{-1}(\mu(P)) = \{XP - PX \mid X \in Ad\,(A)(\mathfrak{so}(2n))\}\,;$$

as for the standard fibre of the Twistor Spaces, we introduce a complex structure on $T_P\mu^{-1}(\mu(P))$, by setting

$$J[P](X) := PX$$

and so μ gives rise to a complex foliation of $Z(n,k)$. Note that $Z(n,k)$ itself can be endowed with a complex structure for which the leaves of the previous foliation are complex submanifolds; in fact, let $\mathbf{J} : \mathfrak{m} \to \mathfrak{m}$ be defined as follows: if $X = \begin{pmatrix} Z & V \\ -{}^tV & 0 \end{pmatrix} \in \mathfrak{m}$, set

$$\mathbf{J}X := \begin{pmatrix} J_n Z & J_n V \\ -{}^t(J_n Y) & 0 \end{pmatrix}.$$

One can check that

i) $\mathbf{J}^2 = -id_{\mathfrak{m}}$;

ii) for every $Y \in \mathfrak{h}$, $ad\,(Y) \circ \mathbf{J} = \mathbf{J} \circ ad\,(Y)$, i. e. for every $X \in \mathfrak{m}$, $[\mathbf{J}X, Y] = \mathbf{J}[X, Y]$;

iii) for every X, $Y \in \mathfrak{m}$, $[\mathbf{J}X, Y] - [X, Y] - \mathbf{J}[\mathbf{J}X, Y] - \mathbf{J}[X, \mathbf{J}Y] \in \mathfrak{h}$.

Therefore, $\mathbf{J}$ defines an integrable invariant almost complex structure on $Z(n,k)$, that coincides with J on the leaves.

We will use these notions to construct the *generalized twistor space*. Let (M,g) be an oriented $2n+k$-dimensional Riemannian manifold and $SO_g(M)$ be the principal $\mathrm{SO}(2n+k)$-bundle of oriented g-orthonormal frames on M; define

$$F^{(n)}(M) := \frac{\mathrm{SO}_g(M)}{\mathrm{U}(n) \times \mathrm{SO}(k)};$$

therefore $F^{(n)}(M)$ is a bundle over M with structure group $\mathrm{SO}(2n+k)$ and standard fibre $Z(n,k)$. Let $r : F^{(n)}(M) \to M$ be the bundle projection and $P \in F^{(n)}(M)$, with $r(P) = x$: P represents an oriented $2n$-dimensional subspace D_x of T_xM, together with a positively oriented g_x-orthogonal complex structure on it. By the previous considerations, $F^{(n)}_x(M) := r^{-1}(x) \simeq Z(n,k)$ is complex foliated by $Z(n,0)$.

REMARK 1.1. The standard fibre $Z(n,k)$ parametrizes the complex structures on the $2n$-dimensional linear subspaces of $\mathbb{R}^{2n+k}$, in fact if

$$T = \begin{pmatrix} J_n & 0 \\ 0 & I_k \end{pmatrix} \text{ and } P = \begin{pmatrix} J_n & B \\ 0 & A \end{pmatrix}$$

give the same complex structure on $\mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n+k}$, then there exists $G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SO}(2n+k)$ with $GT{}^tG = P$. Therefore we have

$$\alpha \in \mathrm{U}(n),\ \beta = 0,\ \gamma = 0,\ \delta \in \mathrm{SO}(k)$$

and then

$$B = 0,\ A = I_k\,.$$

The Levi Civita connection ω on $\mathrm{SO}_g(M)$ induces a splitting on $T_P F^{(n)}(M) \forall\, P \in F^{(n)}(M)$, as

$$T_P F^{(n)}(M) = H_P \oplus W_P$$

into horizontal and vertical parts. By defining $D_P := r_*^{-1}(D_{r(P)})$, $Z_P = T_P Z(n,0)$ and taking the natural metric on $F^{(n)}(M)$ induced by g and ω, we have a further splitting of H_P and W_P:

i) $H_P = D_P \oplus ((D_P)^\perp \cap H_P)$

ii) $W_P = Z_P \oplus ((Z_P)^\perp \cap W_P)\,.$

Consequently

$$T_P F^{(n)}(M) = D_P \oplus Z_P \oplus (D_P \oplus Z_P)^\perp\,. \tag{1.1}$$

Again by the previous considerations $D_P \oplus Z_P$ is endowed with a natural almost complex structure that we will denote by $\mathbb{J}$ and so (1.1) defines an almost complex distribution on $F^{(n)}(M)$.
In the sequel we will denote by $D \oplus Z$ the set $\{X \in\ ,\ (M, TF^{(n)}(M)) : X(P) \in D_P \oplus Z_P\ \forall\, P\}$.

2. Foliations in $F^{(n)}(M)$

It is natural to investigate the conditions in order that the distribution $D \oplus Z$ in $F^{(n)}(M)$ is integrable and, in such a case, the leaves are holomorphic. In this Section we study the first problem. Concerning this, we recall the

FROBENIUS THEOREM: *let P be a C^r $r \geq 1$ k-plane field defined on M. Then P is completely integrable if and only if it is involutive. Further, if either these conditions hold, the leaf tangent to P is unique.*

As specified in (1.1), for $X \in T_P F^{(n)}(M)$, we will denote by $X^{\perp}$, the component of X orthogonal to $D_P \oplus Z_P$ with respect to the Riemannian metric induced by g and ω. With these notations, the involutivity of the distribution is equivalent to the vanishing of the following map

$$\begin{array}{rccl} \psi: & (D \oplus Z) \times (D \oplus Z) & \to & (D \oplus Z)^{\perp} \\ & (X, Y) & \mapsto & [X, Y]^{\perp}\,. \end{array}$$

It is immediate to check that ψ is tensorial: in fact

$$\begin{aligned} \psi(fX, Y) := [fX, Y]^{\perp} &= f[X, Y]^{\perp} - Y(f)X^{\perp} = \\ &= f[X, Y]^{\perp} = f\psi(X, Y)\,. \end{aligned}$$

REMARK 2.1. Let $\nabla^{F^{(n)}}$ be the Levi Civita connection on $F^{(n)}$ and

$$\begin{array}{rccl} \alpha: & D \oplus Z \times D \oplus Z & \to & (D \oplus Z)^{\perp} \\ & (X, Y) & \mapsto & (\nabla_X^{F^{(n)}} Y)^{\perp}\,. \end{array}$$

Immediately we get that the vanishing of α implies the vanishing of ψ, but the converse does not hold.

Then we study the involutivity of $D \oplus Z$.

THEOREM 2.2. *The complex distribution $D \oplus Z$ is involutive if and only if, for every X, $Y \in D$,*

$$(p_*(\Omega(\hat{X}, \hat{Y})^*))^{\perp} = 0\,,$$

where $\hat{X}, \hat{Y}$ are vector fields on $SO_g(M)$ whose projection, induced by $p : SO_g(M) \to F^{(n)}(M)$, is X, Y respectively and Ω is the curvature form of ω.

Proof. We have to consider the following cases:
i) let P be in $F^{(n)}(M)$ and X, Y be in D. Take $\hat{X}$, $\hat{Y}$ vector fields in $SO_g(M)$ such that

a) $\hat{X}, \hat{Y}$ are horizontal with respect to ω;
b) $p_*(\hat{X}) = X$, $p_*(\hat{Y}) = Y$.

We have

$$\begin{aligned}\psi[P](X,Y) &= [X,Y]^{\perp}(P) = [p_*(\hat{X}), p_*(\hat{Y})]^{\perp}(u)\\ &= (p_*[u][\hat{X},\hat{Y}])^{\perp}(u)\ ,\end{aligned}$$

where $p(u) = P$.
Fix $u_0 \in p^{-1}(P)$, let ξ, η be vectors in $\mathbb{R}^{2n+k}$ such that the standard horizontal vector fields $B(\xi)$, $B(\eta)$ on $SO_g(M)$ satisfy $B(\xi)(u_0) = \hat{X}(u_0)$, $B(\eta)(u_0) = \hat{Y}(u_0)$.
Since the form ψ is tensorial,

$$\begin{aligned}\psi[P](X,Y) &= (p_*[u_0][B(\xi), B(\eta)](u_0))^{\perp} =\\ &= -2(p_*[u_0]((\Omega[u_0](B(\xi), B(\eta)))^*(u_0))^{\perp},\end{aligned}$$

that ends the first case.
ii) Let X be in D and Y be in Z. By repeating the same arguments, we may suppose that

$$X(P) = p_*[u_0](\hat{X})(u_0)\ ,\ Y(P) = p_*[u_0](A^*)(u_0)\,,$$

where $\hat{X}$, A^* are a horizontal lift of an appropriately choosen vector field on M and a fundamental vertical vector field on $SO_g(M)$ respectively. Therefore,

$$\psi[u_0](X,Y) = 0\,.$$

iii) By the characterization of $T_P Z(n,p)$ we may consider

$$X(P) = [A_X, P]\frac{\partial}{\partial P} := A_X\hat{}(P)$$

with

$$A_X \in \begin{pmatrix} \mathfrak{so}(2n) & 0 \\ 0 & 0 \end{pmatrix}.$$

$X(P)$ is the fundamental vertical vector field associated to A_X. Therefore

$$[X,Y](P) = [[A_X, P]\frac{\partial}{\partial P}, [A_Y, P]\frac{\partial}{\partial P}] = -[[A_X, A_Y], P]\frac{\partial}{\partial P}$$

which implies the involutivity of the vertical part of $D \oplus Z$. □

REMARK 2.3. Let x_0 be a fixed point of M, $u_0 \in \pi^{-1}(x_0)$, $\pi : SO_g(M) \to M$ being the projection and

$$\{\vartheta_1, \dots, \vartheta_{2n}, \vartheta_{2n+1}, \dots, \vartheta_{2n+k}\}$$

be orthonormal vector fields on M such that $\vartheta_i(x_0) = \dfrac{\partial}{\partial x_i}(x_0)$, for $i = 1, \dots, 2n+k$, where $(x_1, \dots, x_{2n+k})$ are normal coordinates and $p_*[u_0]\vartheta_1^*, \dots, p_*[u_0]\vartheta_{2n+k}^*$ span the vector space $D_{p(u_0)}$, ϑ_i^* being the horizontal lifts of ϑ_i, for $i = 1, \dots, 2n+k$. From now on, we will assume that the Latin indeces run through $1, \dots, 2n$, the Greek through $2n, \dots, 2n+k$ and the capital letters through $1, \dots, 2n+k$. Since the involutivity condition of Theorem 2.2 is tensorial, we may check the integrability condition for $D \oplus Z$ in $p(u_0)$ by taking the vector fields $\tilde{\vartheta}_1, \dots, \tilde{\vartheta}_{2n+k}$ that coinciding with $p_*[u_0]\vartheta_1^*, \dots,$ $p_*[u_0]\vartheta_{2n+k}^*$ at $p(u_0)$.
Therefore, by recalling the expression of the bracket between two horizontal lifts on $SO_g(M)$, we get

$$\begin{aligned}
[\tilde{\vartheta}_i, \tilde{\vartheta}_j]^{\perp}(p(u_0))) &= [p_*\vartheta_i^*, p_*\vartheta_j^*]^{\perp}(p(u_0)) = (p_*[\vartheta_i^*, \vartheta_j^*])^{\perp}(p(u_0)) = \\
&= (p_*(\sum_{k=1}^{2n+k} (\Gamma_{ij}^k - \Gamma_{ji}^k)\vartheta_k - \sum_{s,k,h=1}^{2n+k} R_{kij}^h X_s^k \frac{\partial}{\partial X_s^h}))^{\perp}(p(u_0)) \\
&= (-p_*([R_{\cdot ij}^{\cdot}, X])\frac{\partial}{\partial X})^{\perp}(p(u_0)),
\end{aligned}$$

since we have chosen normal coordinates. Then the conditions is

$$(p_*([R^{\cdot}_{\cdot ij}, X])\frac{\partial}{\partial X})^{\perp}(p(u_0)) = 0\,.$$

3. Complex Foliations in $F^{(n)}(M)$

In this Section we study the conditions ensuring that the almost complex foliation $D \oplus Z$ is complex, i. e. the leaves are complex submanifolds. Let S be an endomorphism of TM: we will denote by $\hat{,}$ the operator

$$S \mapsto S\hat{}(Q) = [S, Q]\frac{\partial}{\partial Q}\,.$$

Remark 3.1. If $A_X\hat{}$ is a fundamental vertical vector field on $SO_g(M)$ and $\tilde{\vartheta}_i$ is a horizontal lift on $SO_g(M)$ as in Remark 2.3, we get

$$\begin{aligned}[A_X\hat{,}\tilde{\vartheta}_i] &= [[A_X\hat{,}P]\frac{\partial}{\partial P}, \vartheta_i - [A_{\Gamma^{\cdot}_{i\,\cdot}}\hat{,}P]\frac{\partial}{\partial P}] = \\ &= -\vartheta_i([A_X\hat{,}P])\frac{\partial}{\partial P} - [A_X\hat{,}A_{\Gamma^{\cdot}_{i\,\cdot}}\hat{}]\frac{\partial}{\partial P} = \\ &= -\vartheta_i(A_X)\hat{} - [A_X\hat{,}A_{\Gamma^{\cdot}_{i\,\cdot}}\hat{}]\frac{\partial}{\partial P}\end{aligned}$$

and, by recalling that $\mathbb{J}$ is the natural almost complex structure on $D \oplus Z$, we have

$$\begin{aligned}\mathbb{J}[P](A_X\hat{}(P)) &= P(A_X\hat{}(P)) = P((A_X\,P - PA_X)\frac{\partial}{\partial P}) = \\ &= (PA_XP - PPA_X)\frac{\partial}{\partial P} = (PA_X)\hat{,}(P)\,.\end{aligned}$$

We have the following

Lemma 3.2. *The Nijenhuis tensor N of the almost complex structure $\mathbb{J}$ on $D \oplus Z$ satisfies*

$$\begin{array}{lll} i) & N(P)(X,Y) = 0 & X, Y \in Z \\ ii) & N(P)(X,\eta) = 0 & X \in Z,\, \eta \in D \\ iii) & N(P)(\xi,\eta) \in T_P Z(n,k) & \xi,\, \eta \in D\,. \end{array}$$

Proof. i) For the integrability of the vertical almost complex structure, we have

$$N(X,Y)=0 \quad \forall X,\, Y\in Z\,.$$

ii) For the tensoriality, we may suppose X fundamental vertical, i. e.

$$X(P)=[A_X,P]\frac{\partial}{\partial P}=A_X\hat{}(P)$$

and

$$\eta=\tilde{\vartheta}_i(P)=p_*[u_0]\vartheta_i^*(u_0)\quad 1\le i\le 2n\,,$$

where $u_0\in p^{-1}(P)$ and the vector fields ϑ_i^* are chosen as in Remark 2.3. By the expression of ϑ_i^* in the trivialization, we get

$$p_*\vartheta_i^*=\vartheta_i-[A_{\Gamma_{i\,\cdot}},P]\frac{\partial}{\partial P}\,.$$

Therefore

$$\begin{aligned}
N_P(X,\tilde{\vartheta}_i) &= [\mathbb{J}(A_X)\hat{},\mathbb{J}\tilde{\vartheta}_i]-[(A_X)\hat{},\tilde{\vartheta}_i]+\\
&\quad -\mathbb{J}[(A_X)\hat{},\mathbb{J}\tilde{\vartheta}_i]-\mathbb{J}[\mathbb{J}(A_X)\hat{},\tilde{\vartheta}_i]=\\
&= [((PA_X))\hat{},P_i^C\tilde{\vartheta}_C]-[(A_X)\hat{},\tilde{\vartheta}_i]+\\
&\quad -\mathbb{J}[(A_X)\hat{},P_i^C\tilde{\vartheta}_C]-\mathbb{J}[(PA_X)\hat{},\tilde{\vartheta}_i]=,
\end{aligned}$$

since $\mathbb{J}[P](A_X\hat{}(P))=(PA_X)\hat{}(P)$,

$$\begin{aligned}
= \;& (PA_X)\hat{}(P_i^l)\tilde{\vartheta}_l-P_il(\vartheta_l(PA_X)+[,\,_l,PA_X])\hat{}+\\
&+(\vartheta_i(X)+[,\,_i,X])\hat{}+\mathbb{J}(\vartheta_i(PA_X)+[,\,_i,PA_X])\hat{}+\\
&+\mathbb{J}P_i^l(\vartheta_l(A_X)+[,\,_l,A_X])\hat{}-\mathbb{J}\,A_X\hat{}(P_i^l)\tilde{\vartheta}_l\,.
\end{aligned}$$

By choosing normal coordinates around $x=r(P)\in M$ in such a way that the vectors $\vartheta_i(x)$ coincide with $\frac{\partial}{\partial x_i}(x)$ $i=1,\ldots,2n$, since P defines a complex structure on Span $\{\tilde{\vartheta}_1(P),\ldots,\tilde{\vartheta}_{2n}(P)\}$, the last expression vanishes at P.

iii) We observe that

$$\begin{aligned}
[\tilde{\vartheta}_i,\tilde{\vartheta}_j] &= p_*[\vartheta_i^*,\vartheta_j^*]=\\
&= p_*((,\,_{ij}^C-,\,_{ji}^C)\vartheta_C-R_{kij}^hX_s^k\tfrac{\partial}{\partial X_s^h})=\\
&= (,\,_{ij}^C-,\,_{ji}^C)\tilde{\vartheta}_C-(R_{\cdot ij}^{\cdot})\hat{}.
\end{aligned}$$

Therefore,

$$\begin{aligned} N_P(\tilde{\vartheta}_i\tilde{\vartheta}_j) &= [\mathbb{J}\tilde{\vartheta}_i, \mathbb{J}\tilde{\vartheta}_j] - [\tilde{\vartheta}_i, \tilde{\vartheta}_j] + \\ &\quad -\mathbb{J}[\tilde{\vartheta}_i, \mathbb{J}\tilde{\vartheta}_j] - \mathbb{J}[\mathbb{J}\tilde{\vartheta}_i, \tilde{\vartheta}_j] = \\ &= [P_i^C\tilde{\vartheta}_C, P_i^D\tilde{\vartheta}_D] - ((,\,{}_{ij}^{C} - ,\,{}_{ji}^{C})\tilde{\vartheta}_C - (R^{\cdot}_{\cdot ij})\hat{}\,) \\ &\quad -\mathbb{J}[\tilde{\vartheta}_C, P_j^D\tilde{\vartheta}_D] - \mathbb{J}[P_i^C\tilde{\vartheta}_C, \tilde{\vartheta}_j] =, \end{aligned}$$

by taking normal coordinates around $r(P)$ and recalling that any point $P \in Z(n,p)$ represents an oriented $2n$-dimensional linear subspace of $\mathbb{R}^{2n+k}$ equipped with a positively oriented orthogonal complex structure, we have $\mathbb{J}[P]\tilde{\vartheta}_i = \sum_{l=1}^{2n} P_i^l\tilde{\vartheta}_l \forall 1 \leq i \leq 2n$ and, then

$$\begin{aligned} &= P_i^l P_j^m[\tilde{\vartheta}_l, \tilde{\vartheta}_m] + P_i^l\tilde{\vartheta}_l(P_j^m)\tilde{\vartheta}_m + \\ &\quad -P_j^m\tilde{\vartheta}_m(P_i^l)\tilde{\vartheta}_l + (R^{\cdot}_{\cdot ij})\hat{}\, + \\ &\quad -\mathbb{J}P_j^l[\tilde{\vartheta}_i, \tilde{\vartheta}_l] - \mathbb{J}(\tilde{\vartheta}_i(P_j^l)\tilde{\vartheta}_l) + \\ &\quad -\mathbb{J}P_i^l[\tilde{\vartheta}_l, \tilde{\vartheta}_j] + \mathbb{J}(\tilde{\vartheta}_j(P_i^l)\tilde{\vartheta}_l) = \\ &= -P_i^l P_j^m\ (R^{\cdot}_{\cdot lm})\hat{}\, + \ (R^{\cdot}_{\cdot ij})\hat{}\, + \\ &\quad +PP_j^l\ (R^{\cdot}_{\cdot il})\hat{}\, + PP_i^l\ (R^{\cdot}_{\cdot lj})\hat{}\,. \end{aligned}$$

Hence, the Lemma is proved. □

4. Holomorphic sections

Let $\tau : M \to F^{(n)}(M)$ be a section of $F^{(n)}(M)$; from the definition of the generalized Twistor Space, τ induces an almost complex $2n$-dimensional distribution on M. We start with the following

PROPOSITION 4.1. *If $\tau : M \to F^{(n)}(M)$ is a section and the distribution $D \oplus Z$ in $F^{(n)}(M)$ is involutive, then the induced distribution, D in M, is also involutive.*

Proof. Let ψ_M be the map

$$\begin{aligned} \psi_M : \quad D \times D &\to \quad D^{\perp} \\ (X, Y) &\mapsto [X, Y]^{\perp}, \end{aligned}$$

where $D^{\perp}$ is the orthogonal complement of D with respect to the metric g. As we have remarked in Section 2, the vanishing of the tensor ψ_M is equivalent to the involutivity of D. Let $\vartheta_1, \ldots, \vartheta_{2n}$ be a local system of generators of the distribution D in M, $\hat{\vartheta_1}, \ldots, \hat{\vartheta_{2n}}$ the horizontal lifts to $SO_g(M)$ and $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_{2n}$ be their projection on $F^{(n)}(M)$. At the points $\tau(x)$, we note that

a) $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_{2n}$ generate the horizontal part of the distribution $D \oplus Z$;

b) $r_*[\tau(x)](\tilde{\vartheta}_i(\tau(x))) = \vartheta_i(x)\ \forall i = 1 \ldots, 2n$.

Since $r_{*_{|H_P}} : H_P \to T_{r(P)}M$ is an isometry, for the vector fields previous defined, we have

$$r_*[\tau(x)]([\tilde{\vartheta}_i, \tilde{\vartheta}_j]^{\perp}(\tau(x))) = [\vartheta_i, \vartheta_j]^{\perp}(x)\,.$$

Therefore the involutivity of $D \oplus Z$ implies the involutivity of D in M. □

In the hypothesis of the last Proposition, M is foliated and the leaves $\mathcal{D}_x$ are almost complex. We will denote by J the almost complex structure on the leaves.

REMARK 4.2. For any almost complex Riemannian manifold (M, g, J), with Levi Civita connection ∇, it is defined a (1,1) tensor field

$$A(X, Y) := (\nabla_{JX} J)Y - J(\nabla_X J)Y\,.$$

The tensor field A has the following properties

$$\begin{aligned} &a) \quad A(X,Y) - A(Y,X) = N_J(X,Y) \\ &b) \quad N_J = 0 \text{ if and only if } A = 0\,. \end{aligned}$$

The following statement characterizes the holomorphic sections.

PROPOSITION 4.3. *Assume that $F^{(n)}(M)$ is almost complex foliated and $\tau : U \to F^{(n)}(M)$ is a local section. The following conditions* *are equivalent*

i) $\tau(U \cap \mathcal{D}_x)$ *is almost complex;*

ii) J *is integrable, i. e. the leaves* $\mathcal{D}_x$ *are complex;*

iii) $\tau(U \cap \mathcal{D}_x)$ *is complex.*

Proof. $i) \Rightarrow ii)$. Let $x \in U$ and $\mathcal{D}_x$ be the leaf through x. We observe that $\tau(U \cap \mathcal{D}_x)$ is an almost complex submanifold of $(\mathcal{D} \oplus \mathcal{Z})_{\tau(x)}$ if and only if $\mathbb{J} \circ \tau_* = \tau_* \circ J$ on $\mathcal{D}_x$. Let $(x^1, \dots, x^{2n+k})$ be normal system of coordinates around x such that $x^{2n+1} = 0, \dots, x^{2n+k} = 0$ define $\mathcal{D}_x$ and $\vartheta_i = \frac{\partial}{\partial x^i}$. We have

$$\begin{aligned} A(\vartheta_i, \vartheta_j) &= (\nabla_{J\vartheta_i} J)\vartheta_j - J(\nabla_{\vartheta_i} J)\vartheta_j = , \\ &\quad \text{by a straightforward computation,} \\ &= \sum_{l,r=1}^{2n} \tau_i^l \vartheta_l(\tau_j^r)\vartheta_r - \sum_{s,r=1}^{2n} \vartheta_i(\tau_j^s)\tau_s^r \vartheta_r \,. \end{aligned} \tag{4.1}$$

Since

$$(\mathbb{J} \circ \tau_*)\vartheta_i = \vartheta_i(\tau_\cdot^s)\tau_s^\cdot \tag{4.2}$$

$$(\tau_* \circ \mathbb{J})\vartheta_i = \tau_i^l \vartheta_l(\tau_\cdot^\cdot)\,, \tag{4.3}$$

then $A(\vartheta_i, \vartheta_j) = 0$ and, by Remark 4.2, it follows that $N_J = 0$.
$ii) \Rightarrow i)$. Let $\Phi_J(X,Y) = g(X, JY)$ be the Kähler form of (M, g, J). Then

$$(\nabla_X \Phi_J)(Y,Z) = -g((\nabla_X J)Y, Z)$$

and

$$\begin{aligned} g(N_J(X,Y),Z) &= -(\nabla_{JX}\Phi_J)(Y,Z) - (\nabla_X \Phi_J)(JY,Z) + \\ &\quad +(\nabla_{JY}\Phi_J)(X,Z) + (\nabla_Y \Phi_J)(JX,Z)\,. \end{aligned}$$

Therefore

$$\begin{aligned} &g(N_J(X,Y),Z) + g(N_J(Z,Y),X) + g(N_J(Z,X),Y) = \\ &= -2g(A(X,Y),Z)\,. \end{aligned}$$

Hence $N_J = 0$ implies $A = 0$. This condition, (4.1), (4.2) and (4.3) give $ii) \Rightarrow i)$.

To prove that $ii) \iff iii)$ it is sufficient to note that $\mathbb{J}_{|_{\tau(U)}} = (\tau^{-1})^*(\tau_*\mathbb{J})$ and, consequently, we have

$$N_{\mathbb{J}}(\cdot,\cdot) = (\tau^{-1})^*(\tau_* N_J)(\cdot,\cdot)\,.$$

□

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